FALL 2022: MATH 790 EXAM 3 SOLUTIONS

Each problem is worth 10 points. You may use your notes from class, your homework or results from Exams 1 and 2, **but you may not use calculators, software or computers of any kind, your book, any other book, topics not explicitly covered in class or homework, the internet, or consult with any students or professors, other than your Math 790 professor**. All details must be provided to receive full credit. Please upload a **pdf file** containing your solutions to the exam to Canvas by 5pm on Wednesday, December 14.

1. Let $A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 2 & 0 & 0 & -1 \\ -1 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$ and define $T : \mathbb{R}^4 \to \mathbb{R}^4$ by $T(v) := A \cdot v$. Find $\mu_T(X)$ and find a maximal vector v_0 for $\mathbb{R}^4$ with respect to T. Then find a T-invariant complement of $\langle T, v_0 \rangle$.

Solution. This problem turned out somewhat easier than I expected. Since $\chi_A(X) = (X+1)(X-2)(X^2+1)$, it follows that $\mu_A(X) = \chi_A(X)$. Thus, a maximal vector v_0 will generate the whole space, i.e., in this case $V = \mathbb{R}^4$ is cyclic with respect to T, so that $V = \langle T, v_0 \rangle$ has no non-trivial T-invariant complement.

To find v_0, we must find maximal vectors for the primary components of V with respect to A. One checks that $v_1 := \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$ is a basis for $\ker(A + 1 \cdot I_4)$, the eigenspace associated to -1, while $v_2 := \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ is a basis for $\ker(A - 2 \cdot I_4)$, the eigenspace associated to 2. Moreover, $A^2 + I = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 4 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & 1 \end{pmatrix}$, and one readily checks that $v_3 := \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$ and $v_4 := A \cdot v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$ form a cyclic basis for $\ker(A^2 + I)$. Therefore, $v_0 := v_1 + v_2 + v_3 = \begin{pmatrix} 1 \\ 3 \\ 1 \\ 1 \end{pmatrix}$ is a maximal vector for V with respect to A (or T). One can verify this by showing that the determinant of the matrix with columns $v_0, A \cdot v_0, A^2 \cdot v_0, A^3 \cdot v_0$ is non-zero.

2. Let $A = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ and let $T : \mathbb{Q}^4 \to \mathbb{Q}^4$ be defined by $T(v) = A \cdot v$. Find $\mu_T(x)$ and write $\mathbb{Q}^4$ as a direct sum of cyclic subspaces, first in terms of elementary divisors and then in terms of invariant factors. Then find the corresponding rational canonical forms of T.

Solution. On the one hand, $\chi_A(X) = (X^2+1)^2$, while on the other hand $A^2 + I = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, so $\mu_A(X) = \chi_A(X)$. Thus, in this case, both forms of the rational canonical form are the same, namely the companion matrix of $\mu_A(X)$, $C := \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \end{pmatrix}$. Likewise, it follows that $\mathbb{Q}^4$ is a cylic space with respect to T. To find a cyclic (i.e., maximal) vector v, we must choose a vector v not in $\ker(A^2 + I)$. Note that $v := \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ is not in $\ker(A^2 + I)$ and the vectors v, Av, A^2v, A^3v form a basis for $\mathbb{Q}^4$, since the matrix whose columns are these vectors has non-zero determinant. Thus v is a cyclic vector for $\mathbb{Q}^4$. It is now easy to check that if P is the matrix whose columns are v, Av, A^2v, A^3v, i.e., $P = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -2 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}$, then $P^{-1}AP = C$.

3. For the matrix $A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ -3 & -2 & 0 & 2 \end{pmatrix}$, find $\mu_A(x)$ and then find, with proof, invertible 4×4 matrices P and Q over $\mathbb{R}$ such that $P^{-1}AP$ is in rational canonical form and $Q^{-1}AQ$ is in Jordan canonical form.

Solution. One checks that $\chi_A(X) = (X-2)^4$, while $\mu_A(X) = (X-2)^3 = X^3 - 6X^2 + 12X - 8$. Since $\mu_A(X)$ has one irreducible factor, the invariant factor rational canonical form of A equals the elementary divisor rational canonical form of A. Since the companion matrix of $\mu_A(X)$ is a 3×3 matrix, the remaining 1×1 block must be the companion matrix of $X - 2$. Therefore the RCF of A is $B := \begin{pmatrix} 0 & 0 & 8 & 0 \\ 1 & 0 & -12 & 0 \\ 0 & 1 & 6 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$.

To find P, we must find a basis corresponding to a cyclic decomposition of $\mathbb{R}^4$ with respect to A. For this, we must find a vector v not in $\ker(A-2I)^2$. Since $(A-2I)^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \end{pmatrix}$, it is easy to see that e_2, e_3, e_4 form a basis for $\ker(A-2I)^2$, so $v = e_1$ is a maximal vector for $\mathbb{R}^4$ with respect to A. Here the e_i are just the standard basis elements. Thus, we take for the first three columns of P, the vectors v, Av, A^2v. For the fourth vector, we must take a vector in the eigenspace of A associated to the value 2, and one easily sees that the vector e_4 works. Thus, we have $P = \begin{pmatrix} 1 & 2 & 4 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -3 & -14 & 1 \end{pmatrix}$. A quick check shows $A \cdot P = P \cdot B$, which is what we want.

For the Jordan form J of A, we note that since $\mu_A(X) = (X-2)^3$, J must have a 3×3 Jordan block associated to 2. Since this leaves just a 1×1 block, that block must just be 2. Therefore the Jordan form of A is $J = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$. Since $\mu_A(X) = (X-2)^3$, we have the same maximal vector as above, namely, $v = e_1$, and thus $v, (A-2I)v, (A-2I)^2v, e_4$ will given the Jordan decomposition we want. That is, if we take Q to be the matrix with columns $v, (A-2I)v, (A-2I)^2v, e_4$, then $Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -3 & -2 & 1 \end{pmatrix}$ and $Q^{-1}AQ = J$.

4. Let $T \in \mathcal{L}(V)$ and assume that the minimal polynomial of T has all of its roots in F. Prove that the following statements are equivalent.

(i) There do not exist non-trivial T-invariant subspaces $U, W \subseteq V$ such that $V = U \oplus W$.
(ii) The Jordan form of T consists of a single $n \times n$ Jordan block.

Solution. Suppose the Jordan form of T consists of a single block. If we could write $V = U \bigoplus W$, with U and W T-invariant subspaces, then any matrix of T with respect to a basis obtained by putting together bases for U and W would have two blocks. In particular, since the minimal polynomials for $T|_U$ and $T|_W$ divide the minimal polynomial for T, they each have their roots in F. Thus, there is a basis B_U for U and a basis B_W for W such that the matrices of $T|_U$ and $T|_W$ with respect to these bases are in Jordan form. But then the matrix of T with respect to $B_U \cup B_W$ is also in Jordan form, with at least two Jordan blocks. Since Jordan forms are unique, up to a permutation of Jordan blocks, we have a contradiction. Thus, we cannot write $V = U \bigoplus W$, with both U and W T-invariant subspaces of V.

Conversely, suppose the Jordan form of T has more than one block. By gathering together multiple blocks, we can assume there is a matrix for T that has just two blocks. Let $\mathcal{B}$ denote a basis for V such that the matrix of T with respect to $\mathcal{B}$ has just two blocks. If the first block of this matrix occupies columns $1, 2, \ldots, c$ and the second block occupies columns $c+1,\dots,n$, let U denote the space spanned by the first c vectors in $\mathcal{B}$ and W denote the space spanned by the remaining $c-n$ elements of $\mathcal{B}$. Thus, $V = U\bigoplus W$. On the other hand, by definition, T carries the basis elements of U into U and the basis elements of W into W, and thus U and W are T-invariant subspaces.

5. Find three distinct cube roots of the matrix $A = \begin{pmatrix} -1 & 0 & 0\\ 2 & -16 & -25\\ -1 & 9 & 14\end{pmatrix}$. Then find A^{2022}.

Solution. We first note that the minimal polynomial if A is $(x+1)^3$, and the JCF of A is $J := \begin{pmatrix} -1 & 0 & 0\\ 1 & -1 & 0\\ 0 & 1 & -1\end{pmatrix}$. Moreover, for $P = \begin{pmatrix} 1 & 0 & 0\\ \frac19 & \frac13 & -5\\ 0 & 0 & 3\end{pmatrix}$, $P^{-1} = \begin{pmatrix} 1 & 0 & 0\\ -\frac13 & 3 & 5\\ 0 & 0 & \frac13\end{pmatrix}$, and $A = PJP^{-1}$. Using the inductive definition of $p_n(x)$ given in the lecture of November 16, it is not difficult to see that $p_3(x) = 1 + \frac13 x - \frac19 x^2$ is the required polynomial satisfying $p_3(x)^3 = (1+x) + x^3q(x)$, for $q(x) \in \mathbb{R}[x]]$. SInce $\lambda = -1$ is the eigenvalue, $\lambda^{-1} = -1$, so for $B_0 := \begin{pmatrix} 0 & 0 & 0\\ -1 & 0 & 0\\ 0 & -1 & 0\end{pmatrix}$, $p_3(B_0) = \begin{pmatrix} 1 & 0 & 0\\ -\frac13 & 1 & 0\\ -\frac19 & -\frac13 & 1\end{pmatrix}$, so that $p_3(B_0)^3 = \begin{pmatrix} 1 & 0 & 0\\ -1 & 1 & 0\\ 0 & -1 & 1\end{pmatrix}$. Thus, $J_0 := -p_3(B_0) = \begin{pmatrix} -1 & 0 & 0\\ \frac13 & -1 & 0\\ \frac19 & \frac13 & -1\end{pmatrix}$ satisfies $J_0^3 = J$. Thus, $B := PJ_0P^{-1} = \begin{pmatrix} -1 & 0 & 0\\ \frac19 & -6 & -\frac{25}{3}\\ 0 & 3 & 4\end{pmatrix}$ satisfies $B^3 = A$. Moreover, for $\omega := e^{\frac{2\pi i}{3}}$, ωB and $\omega^2 B$ are also cube roots of A.

We also have $J^{2022} = \begin{pmatrix} 1 & 0 & 0\\ -2022 & 1 & 0\\ \binom{2022}{2} & -2022 & 1\end{pmatrix}$ and $A^{2022} = PJ^{2022}P^{-1} = \begin{pmatrix} 1 & 0 & 0\\ -5c-4044 & 30331 & 505515\\ 3c+2022 & -18198 & -303299\end{pmatrix}$, where $c = \binom{2022}{2}$.

6. Let $A \in \mathrm{M}_n(\mathbb{C})$. Prove that $\det(e^A) = e^{\mathrm{trace}(A)}$.

Solution. Let J denote the JCF of A and P satisfy $A = PJP^{-1}$. Then $e^A = Pe^JP^{-1}$. Therefore, $\det(e^A) = \det(e^J)$. If we let $\lambda_1,\dots,\lambda_n$ denote the (not necessarily distinct) eigenvalues of A, then from class we have seen that e^J is lower triangular with $e^{\lambda_1},\dots,e^{\lambda_n}$ down its diagonal. Thus,

$$\det(e^A) = \det(e^J) = e^{\lambda_1}\cdots e^{\lambda_n} = e^{\lambda_1+\cdots+\lambda_n} = e^{\mathrm{trace}(J)}.$$

To see that $\mathrm{trace}(J) = \mathrm{trace}(A)$, one uses the (easy to prove) fact $\mathrm{trace}(MN) = \mathrm{trace}(NM)$, for matrices M and N, together with $A = PJP^{-1}$.

7. Let U, V, W be vector spaces over the field F. Prove $U\otimes(V\oplus W) \cong (U\otimes V)\oplus(U\otimes W)$. Here, $\oplus$ denotes the external direct sum.

Solution. We seek linear transformations

$$T : U\otimes(V\oplus W) \to (U\otimes V)\oplus(U\otimes W) \text{ and } S : (U\otimes V)\oplus(U\otimes W) \to U\otimes(V\oplus W)$$

such that the two compositions ST and TS give the identity transformations.

Let $\psi_0 : U\times(V\oplus W) \to (U\otimes V)\oplus(U\otimes W)$ be defined by $\psi_0(u,(v,w)) = (u\otimes v, u\otimes w)$. Then ψ_0 is easily seen to be a bilinear map. We check just one of the conditions to verify the bilinearity. For example,

$$\psi_0(u,(v_1,w_1)+(v_2,w_2)) = \psi_0(u,(v_1+v_2,w_1+w_2)) = (u\otimes(v_1+v_2), u\otimes(w_1+w_2))$$
$$= (u\otimes v_1 + u\otimes v_2, u\otimes w_1 + u\otimes w_2) = (u\otimes v_1, u\otimes w_1) + (u\otimes v_2, u\otimes w_2)$$
$$= \psi_0(u,(v_1,w_1)) + \psi_0(u,(v_2,w_2)).$$

Thus, there exists $T : U\otimes(V\oplus W) \to (U\otimes V)\oplus(U\otimes W)$ such that $T\circ\phi = \psi_0$, where $\phi(u,(v,w)) = u\otimes(v,w)$, for $(u,(v,w)) \in U\times(V\oplus W)$ is the bilinear map accompanying $U\otimes(V\oplus W)$. In particular, $T(u\otimes(v,w)) = (u\otimes v, u\otimes w)$, for all $u\otimes(v,w) \in U\otimes(V\oplus W)$.

To find $S : (U\otimes V)\oplus(U\otimes W) \to U\otimes(V\oplus W)$, we will find linear transformations $S_1 : U\otimes V \to U\otimes(V\oplus W)$ and $S_2 : (U\otimes W) \to U\otimes(V\oplus W)$ satisfying $S_1(u\otimes v) = u\otimes(v,0)$ and $S_2(u\otimes w) = u\otimes(0,w)$, for all $u\otimes v \in U\otimes V$ and $u\otimes w \in U\otimes W$. Suppose S_1 and S_2 exist. Then we define $S : (U\otimes V)\oplus(U\otimes W) \to U\otimes(V\oplus W)$ by $S(u\otimes v, u'\otimes w) := S_1(u'\otimes v) + S_2(u\otimes w)$. Note that S is well-defined and we have

$$ST(u\otimes(v,w)) = S(u\otimes v, u\otimes w) = S_1(u\otimes v) + S_2(u\otimes w) = u\otimes(v,0) + u\otimes(0,w) = u\otimes(v,w).$$

Similarly,

$$TS(u \otimes v, u' \otimes w) = T(S_1(u \otimes v) + S_2(u' \otimes w)) = T(u \otimes (v,0) + u' \otimes (0,w))$$
$$= T(u \otimes (v,0)) + T(u' \otimes (0,w)) = (u \otimes v, u \otimes 0) + (u' \otimes 0, u' \otimes w) = (u \otimes v, u' \otimes w),$$

since in any tensor product, $a \otimes 0 = 0 = 0 \otimes b$. Thus, ST and TS are the identity transformations, as required.

To find S_1, consider $\psi' : U \times V \to U \otimes (V \oplus W)$ given by $\psi'(u,v) = u \otimes (v,0)$. It is straight forward to check that ψ' is bilinear, so there exists $S_1 : U \otimes V \to U \otimes (V \oplus W)$ such that $T \circ \phi' = \psi'$, where $\phi'(u,v) = u \otimes v$ is the given bilinear map from $U \times V \to U \otimes V$. Thus, $S_1(u \otimes v) = u \otimes (v,0)$, for all $u \otimes v$, as required. The required S_2 is constructed similarly.

8. Let $S : U_1 \to U_2$ and $T : V_1 \to V_2$ be linear transformations. Show that there exists a unique linear transformation $S \otimes T : U_1 \otimes V_1 \to U_2 \otimes V_2$ satisfying $(S \otimes T)(u \otimes v) = S(u) \otimes T(v)$, for all $u \otimes v \in U_1 \otimes V_1$.

Solution. For this, we define $\psi : U_1 \times V_1 \to U_2 \otimes V_2$ by $\psi(u,v) := S(u) \otimes T(v)$, for all $(u,v) \in U_1 \times V_1$. Then it is easy to check that ψ is bilinear. For example,

$$\psi(u_1 + u_2, v) = S(u_1 + u_2) \otimes T(v) = (S(u_1) + S(u_2)) \otimes T(v) = S(u_1) \otimes T(v) + S(u_2) \otimes T(v) = \psi(u_1,v) + \psi(u_2,v).$$

Now, let $\phi : U_1 \times V_1 \to U_1 \otimes V_1$ be the given bilinear map that takes (u,v) to $u \otimes v$. Then there exists a unique linear transformation $f : U_1 \otimes V_1 \to U_2 \otimes V_2$ such that $f \circ \phi(u,v) = \psi(u,v)$, i.e., $f(u \otimes v) = S(u) \otimes T(v)$, for all $u \otimes v \in U \otimes V$. This gives what we want, and we write $S \otimes T$ instead of f.

9. Let $S \in \mathcal{L}(U,U)$ and $T \in \mathcal{L}(V,V)$, where U and V are finite dimensional vectors spaces over the field F. Let $B_U := \{u_1, \ldots, u_m\}$ be a basis for U and $B_V := \{v_1, \ldots, v_n\}$ be a basis for V. Set $A := [S]_{B_U}^{B_U}$ and $B := [T]_{B_V}^{B_V}$. Set

$$B_{U \otimes V} := \{u_1 \otimes v_1, \ldots, u_1 \otimes v_n, u_2 \otimes v_1, \ldots, u_2 \otimes v_n, \ldots, u_m \otimes v_1, \ldots, u_m \otimes v_n\}$$

and set $A \otimes B := [S \otimes T]_{B_{U \otimes V}}^{B_{U \otimes V}}$. Describe the matrix $A \otimes B$ in terms of the matrices A and B. Then prove $\det(A \otimes B) = \det(A)^n \cdot \det(B)^m$ and $\mathrm{trace}(A \otimes B) = \mathrm{trace}(A) \cdot \mathrm{trace}(B)$. Hint: First show $S \otimes T = (S \otimes id_V) \circ (id_U \otimes T)$.

Solution. Straightforward calculation shows that $A \otimes B$, the matrix of $S \otimes T$ with repsect to the given basis is an $mn \times mn$ matrix that can be subdivided into n^2 $m \times m$ blocks as follows:

$$A \otimes B = \begin{pmatrix} a_{11} \cdot B & a_{12} \cdot B & \cdots & a_{1m} \cdot B \\ \vdots & \vdots & \ldots & \vdots \\ a_{m1} \cdot B & a_{m2} \cdot B & \cdots & a_{mm} \cdot B \end{pmatrix}.$$

It follows immediately, that $\mathrm{trace}(A \otimes B) = \mathrm{trace}(A) \cdot \mathrm{trace}(B)$. Moroever, for any $u \otimes v \in U \otimes V$,

$$(S \otimes id_V) \circ (id_U \otimes T)(u \otimes v) = (S \otimes id_V)(u \otimes T(v) = S(u) \otimes T(v).$$

Thus, if $C := [S \otimes id_V]_{B_{U \otimes V}}^{B_{U \otimes V}}$ and $D := [id_U \otimes T]_{B_{U \otimes V}}^{B_{U \otimes V}}$, then $A \otimes B = C \cdot D$, and hence $\det(A \otimes B) = \det(C) \cdot \det(D)$. Since $C = I_m \otimes B$, C is block diagonal, with m blocks of B down the diagonal, and thus $\det(C) = \det(B)^m$. On the other hand,

$$D = A \otimes I_n = \begin{pmatrix} a_{11} \cdot I_n & a_{12} \cdot I_n & \cdots & a_{1m} \cdot I_n \\ \vdots & \vdots & \ldots & \vdots \\ a_{m1} \cdot I_n & a_{m2} \cdot I_n & \cdots & a_{mm} \cdot I_n \end{pmatrix}.$$

I leave it to you to check that there is a sequence of row operations $E_1, \ldots, E_r$ and an equal number of column operations $F_1, \ldots, F_r$ such that: each E_j is an interchange of two rows, each F_i is an interchange of two columns, and if R_i interchanges rows a and b, then F_i interchanges coulmns a and b, and the resulting matrix D' is block diagonal with n copies of A down the diagonal. Since D' is obtained through an even number of row or column exchnages, $\det(D) = \det(D') = \det(A)^n$. Thus,

$$\det(A \otimes B) = \det(C) \cdot \det(D) = \det(B)^m \cdot \det(A)^n.$$

10. Let V and W be vector spaces. Recall that V^* denotes the dual space of V.

(i) Prove that there exists a unique linear transformation $T : V^* \otimes W \to \mathcal{L}(V,W)$ such that $T(f \otimes w)(v) = f(v)w$, for all $f \otimes w \in V^* \otimes W$ and $v \in V$.

(ii) Prove that if V and W are finite dimensional, then T is an isomorphism.

(iii) Let $V = \mathbb{R}^4$, $W = \mathbb{R}^3$, $f_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix} \in V^*$, $f_2 = \begin{pmatrix} -1 & 0 & 1 & 0 \end{pmatrix} \in V^*$, $w_1 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$, $w_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$. Find the matrix of $T(f_1 \otimes w_1 + f_2 \otimes w_2)$ with respect to the standard bases of V and W.

Solution. For part (i), let $\phi : V^* \times W \to V^* \otimes W$ be the given map. We define $h : V^* \times W \to \mathcal{L}(V, W)$ as $h(f, w) = f_w$, where for all $v \in V$, $f_w(v) := f(v)w$. Since f is linear, it follows immediately that $f_w \in \mathcal{L}(V, W)$, for all $w \in W$ and $f \in V^*$. Now, for $f_1, f_2 \in V^*$ and $w \in W$, $h(f_1 + f_2, w) = (f_1 + f_2)_w$. Thus, for all $v \in V$ we have,

$$\begin{aligned} h(f_1 + f_2, w)(v) &= (f_1 + f_2)(v)w \\ &= (f_1(v) + f_2(v))w \\ &= f_1(v)w + f_2(v)w \\ &= h(f_1, w)(v) + h(f_2, w)(v), \end{aligned}$$

and thus, $h(f_1+f_2, w) = h(f_1, w)+h(f_2, w)$. Similarly, one can show $h(f, w_1+w_2) = h(f, w_1)+h(f, w_2)$. For $\lambda \in F$,

$$h(\lambda f, w)(v) = (\lambda f)(v)w = \lambda\{f(v)w\} = \lambda\{h(f, w)(v)\},$$

showing that $h(\lambda f, w) = \lambda h(f, w)$. Similarly, one can show $h(f, \lambda w) = \lambda h(f, w)$, so that h is bilinear. Thus, there exists a unique linear transformation $T : V^* \otimes W \to \mathcal{L}(V, W)$ so that $T\phi = h$. In other words, $T(f \otimes w) = f_w$, which means $T(f \otimes w)(v) = f_w(v) = f(v)w$, for all $v \in V$.

For part (ii), suppose that $\{v_1, \ldots, v_n\}$ is a basis for V and $\{w_1, \ldots, w_m\}$ is a basis for W. Let $\{v_1^*, \ldots, v_n^*\}$ denote the corresponding dual basis for V^*. Then $\{v_i^* \otimes w_j\}_{1 \le i \le n, 1 \le j \le m}$ is a basis for $V^* \otimes W$. Now, for each v_k, $T(v_i^* \otimes w_j)(v_k) = w_j$, if $k = i$ and $T(v_i^* \otimes w_j)(v_k) = 0$, if $k \neq i$. This shows that $T(v_i^* \otimes w_j)$ is a basis for $\mathcal{L}(V, W)$. In other words, T takes a basis of $V^* \otimes W$ to a basis of $\mathcal{L}(V, W)$, and hence these spaces are isomorphic.

For part (iii), let $E := \{e_1, e_2, e_3, e_4\}$ denote the standard basis of $\mathbb{R}^4$ and U denote the standard basis of $\mathbb{R}^3$. Then,

$$T(f_1 \otimes w_1 + f_2 \otimes w_2)(e_1) = T(f_1 \otimes w_1)(e_1) + T(f_2 \otimes w_2)(e_2) = 1 \cdot w_1 + -1 \cdot w_2 = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix}.$$

$$T(f_1 \otimes w_1 + f_2 \otimes w_2)(e_2) = T(f_1 \otimes w_1)(e_2) + T(f_2 \otimes w_2)(e_2) = 2 \cdot w_1 + 0 \cdot w_2 = \begin{pmatrix} 6 \\ 4 \\ 2 \end{pmatrix}.$$

$$T(f_1 \otimes w_1 + f_2 \otimes w_2)(e_3) = T(f_1 \otimes w_1)(e_3) + T(f_2 \otimes w_2)(e_3) = 3 \cdot w_1 + 1 \cdot w_2 = \begin{pmatrix} 8 \\ 6 \\ 4 \end{pmatrix}.$$

$$T(f_1 \otimes w_1 + f_2 \otimes w_2)(e_4) = T(f_1 \otimes w_1)(e_4) + T(f_2 \otimes w_2)(e_4) = 4 \cdot w_1 + -0 \cdot w_2 = \begin{pmatrix} 12 \\ 8 \\ 4 \end{pmatrix}.$$

Therefore,

$$[T(f_1 \otimes w_1 + f_2 \otimes w_2]_E^U = \begin{pmatrix} 4 & 6 & 8 & 12 \\ 2 & 4 & 6 & 8 \\ 0 & 2 & 4 & 4 \end{pmatrix}.$$

Bonus Problems. Each problem below is worth 5 points. Solutions must be completely correct in order to receive any credit.

(i) Show that a vector space over an infinite field cannot be written as a union of finitely many proper subspaces. Give an example to show this is false over a finite field.

(ii) For vector spaces U, V, W over the field F, prove that $U \otimes (V \otimes W) \cong (U \otimes V) \otimes W$.

(iii) Let $\{f_n\}$ be the Fibonacci sequence $f_1 = 1$, $f_2 = 1$, $f_3 = 2$, $f_4 = 3$, ..., $f_n = f_{n-1} + f_{n-2}$. Prove that for all $n \ge 1$,

$$f_n = \frac{1}{\sqrt{5}}\{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\}.$$

Hint: Write $\begin{pmatrix} f_n \\ f_{n-1} \end{pmatrix} = A^{n-2} \cdot \begin{pmatrix} u \\ v \end{pmatrix}$, for some 2×2 matrix A and $u, v \in \mathbb{Z}$.

(iv) Let A and B be $n \times n$ matrices and set $C := AB - BA$. If $AC = CA$, prove that C is a nilpotent matrix.

(v) Let V and W be vector spaces over $\mathbb{C}$ of dimensions n and m. Set $U := \mathcal{L}(V, W)$. Fix isomorphisms $\alpha \in \mathcal{L}(V, V)$ and $\beta \in \mathcal{L}(W, W)$. Define $\phi : \mathcal{L}(U, U) \to \mathcal{L}(U, U)$ by $\phi(T) = \beta^{-1}T\alpha$, for all $T \in U$. Find formulas for trace(ϕ) and det(ϕ) in terms of α and β.